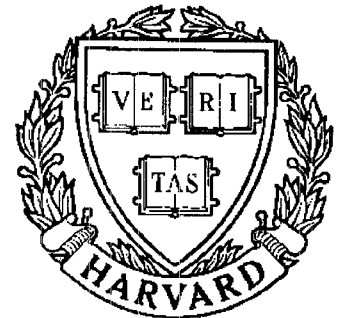


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## **Asymptotic Normality of the Contraction Mapping Estimator for Frequency Estimation**

*by T-H. Li , B. Kedem, and S. Yakowitz*

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# ASYMPTOTIC NORMALITY OF THE CONTRACTION MAPPING ESTIMATOR FOR FREQUENCY ESTIMATION <sup>1</sup>

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## **Abstract**

This paper investigates the asymptotic distribution of the recently-proposed contraction mapping (CM) method for frequency estimation. Given a finite sample composed of a sinusoidal signal in additive noise, the CM method applies to the data a parametric filter that matches its parameter with the first-order autocorrelation of the filtered noise. The CM estimator is defined as the fixed-point of the parametrized first-order sample autocorrelation of the filtered data. In this paper, it is proved that under appropriate conditions, the CM estimator is asymptotically normal with a variance inversely related to the signal-to-noise ratio. A useful example of the AR(2) filter is discussed in detail to illustrate the performance of the CM method.

**Abbreviated Title:** “CM Method for Frequency Estimation”

**Key words and phrases:** Frequency estimation, iterative filtering, normality, parametric filter, spectrum analysis.



# 1 INTRODUCTION

The contraction mapping (CM) method is an iterative filtering approach for frequency estimation on the basis of a noisy sample composed of a sinusoidal signal in additive noise (see He and Kedem, 1990a; Yakowitz, 1991a; Kedem, 1990; Li and Kedem, 1991). On applying to the data a parametric filter that satisfies the so-called fundamental property, i.e., the parameter of the filter matches the first-order autocorrelation of the filtered noise, the CM estimator is obtained by finding a fixed-point of the parametrized first-order sample autocorrelation of the filtered data. In a recent work by Li and Kedem (1991), existence and strong consistency of the CM estimator have been proved based on a general result concerning uniform strong consistency of the parametrized sample autocovariances of the filtered data. It was shown (Li and Kedem, 1991) that under some regularity conditions on the filter and for sufficiently large sample size, the first-order sample autocorrelation forms a contraction mapping in the vicinity of the true frequency, and thus a unique fixed-point can be found with probability one by using standard iterative procedures such as the fixed-point iteration (FPI). Furthermore, as the sample size tends to infinity, the fixed-point constitutes a strongly consistent estimator of the frequency. In this paper we shall show that the CM estimator is also asymptotically normal. We shall also provide a further analysis of the AR(2) filter considered by Li and Kedem (1991), focusing on its asymptotic variance and efficiency. As a special case of the CM method using an AR(2) filter, Quinn and Fernandes (1988) showed that an efficiency as high as that of the nonlinear least squares (NLS) estimator can be achieved in the limiting case as the bandwidth of the general AR(2) filter shrinks toward zero. This, together with the results in this paper, shows the flexibility and efficiency of the CM method for frequency estimation. A numerical example is presented to support the theoretical analysis.

An entirely different proof of the strong consistency of the CM method has been given in Kedem and Yakowitz (1991) using filtered zero-crossing counts (HOC).

This paper is arranged as follows. Section 2 briefly summarizes the CM method and some asymptotic properties obtained by Li and Kedem (1991). In Section 3, general results are given concerning the asymptotic normality of sample autocovariances in the case of a sinusoidal signal in additive *colored* noise. These results generalize those of Mackisack and Poskitt (1989) (see also Stoica *et al.*, 1989) where the asymptotic normality was established only for white noise.

The asymptotic normality of the CM estimator is presented in Section 4 based on the general results obtained in Section 3. Finally, in Section 4, the AR(2) filter is considered in detail as an important example of the CM method. In this section, the asymptotic variance of the CM estimator is derived explicitly for the AR(2) filter, and some numerical simulations are provided to illustrate its performance.

## 2 THE CONTRACTION MAPPING METHOD

Consider the time series  $\{y_t\}$  defined by

$$y_t := \beta \cos(\omega_0 t + \phi) + \epsilon_t \quad (2.1)$$

where  $\beta > 0$  and  $\omega_0 \in (0, \pi)$  are constants, and  $\{\epsilon_t\}$  is a linear process of the form

$$\epsilon_t = \sum_j \psi_j \xi_{t-j} \quad (2.2)$$

with  $\sum |\psi_j| < \infty$ , and  $\{\xi_t\}$  being independent and identically distributed with mean zero and variance  $\sigma_\xi^2$ . The phase  $\phi$  of the sinusoid can be random or constant. For convenience, we assume that  $\phi$  is uniformly distributed on  $(-\pi, \pi]$  and independent of  $\{\xi_t\}$ . Under this assumption,  $\{y_t\}$  becomes wide-sense stationary with mean zero. The frequency estimation problem is to estimate the unknown frequency  $\omega_0$  based on a finite sample of  $\{y_t\}$ .

Before summarizing the CM method, let us introduce some useful notations. Let  $\mathcal{L}_\alpha$  be a linear time-invariant causal filter with real-valued impulse response sequence  $\{h_j(\alpha), j = 0, 1, \dots\}$ , where  $\alpha \in [\underline{\alpha}, \bar{\alpha}]$  with  $\underline{\alpha}$  and  $\bar{\alpha}$  being constants such that

$$-1 < \underline{\alpha} < \cos \omega_0 < \bar{\alpha} < 1.$$

Denote by  $H(\omega; \alpha)$  the transfer function of  $\mathcal{L}_\alpha$ , i.e.,

$$H(\omega; \alpha) := \sum_{j=0}^{\infty} h_j(\alpha) e^{-ij\omega}.$$

Let  $\{y_t(\alpha)\}$  and  $\{\epsilon_t(\alpha)\}$ , defined by

$$y_t(\alpha) := \sum_{j=0}^{\infty} h_j(\alpha) y_{t-j} \quad \text{and} \quad \epsilon_t(\alpha) := \sum_{j=0}^{\infty} h_j(\alpha) \epsilon_{t-j},$$



be the outputs of the filter  $\mathcal{L}_\alpha$  when applied to  $\{y_t\}$  and  $\{\epsilon_t\}$ , respectively. Define

$$\rho_k^\epsilon(\alpha) := \frac{E\{\epsilon_{t+k}(\alpha)\epsilon_t(\alpha)\}}{E\{\epsilon_t^2(\alpha)\}}$$

to be the  $k$ th-order autocorrelation of  $\{\epsilon_t(\alpha)\}$ .

Suppose in the sequel that  $\mathcal{L}_\alpha$  is chosen such that for *any*  $\alpha \in [\underline{\alpha}, \bar{\alpha}]$ , the following fundamental property is satisfied:

$$\alpha = \rho_1^\epsilon(\alpha), \quad (2.3)$$

that is, the parameter  $\alpha$  of the filter matches the first-order autocorrelation of the filtered noise  $\{\epsilon_t(\alpha)\}$ . Under this assumption, the first-order autocorrelation of  $\{y_t(\alpha)\}$ , denoted by  $\rho(\alpha)$ , can be written as

$$\rho(\alpha) = \alpha^* + C(\alpha)(\alpha - \alpha^*) \quad (2.4)$$

where

$$\alpha^* := \cos \omega_0$$

is the parameter to be estimated, and where

$$C(\alpha) := \frac{1}{1 + \gamma(\alpha)}$$

is the so-called contraction factor and  $\gamma(\alpha)$  the signal-to-noise ratio (SNR) of  $\{y_t(\alpha)\}$ , i.e.,

$$\gamma(\alpha) := \frac{\beta^2 |H(\omega_0; \alpha)|^2}{2 E\{\epsilon_t^2(\alpha)\}}.$$

For convenience, let us define  $G(\alpha) := 1 - C(\alpha)$  (known as the gain factor), i.e.,  $G(\alpha) = \gamma(\alpha)/\{1 + \gamma(\alpha)\}$ , then (2.4) can be rewritten as

$$\alpha - \rho(\alpha) = G(\alpha)(\alpha - \alpha^*). \quad (2.5)$$

It is clear that if the filter  $\mathcal{L}_\alpha$  captures the signal (i.e., if  $G(\alpha) > 0$ , or, equivalently,  $C(\alpha) < 1$ ) for all  $\alpha$  in a neighborhood of  $\alpha^*$ , then  $\alpha^*$  would be the unique *fixed-point* of the mapping  $\rho(\alpha)$  in that neighborhood. This gave rise to the idea of the CM method that estimates  $\alpha^*$  by locating the fixed-point of an *estimator* of  $\rho(\alpha)$ .

The CM method can be summarized as follows. For a given sample  $\{y_0, y_1, \dots, y_{n-1}\}$  of size  $n$ , let

$$\hat{y}_t(\alpha) := \sum_{j=0}^t h_j(\alpha) y_{t-j} \quad (t = 0, 1, \dots, n-1).$$

be the filtered data. Take  $\hat{\rho}(\alpha)$ , an estimator of  $\rho(\alpha)$ , to be, for instance,

$$\hat{\rho}(\alpha) := \frac{\hat{r}_1(\alpha)}{\hat{r}_0(\alpha)} \quad (2.6)$$

where

$$\hat{r}_0(\alpha) := n^{-1} \sum_{t=0}^{n-1} \hat{y}_t^2(\alpha) \quad \text{and} \quad \hat{r}_1(\alpha) := n^{-1} \sum_{t=0}^{n-2} \hat{y}_{t+1}(\alpha) \hat{y}_t(\alpha). \quad (2.7)$$

Then, use the so-called fixed-point iteration (FPI)

$$\hat{\alpha}_n^{(m)} := \hat{\rho}(\hat{\alpha}_n^{(m-1)}) \quad (m = 1, 2, \dots) \quad (2.8)$$

to find a fixed-point of  $\hat{\rho}(\alpha)$  in the vicinity of  $\alpha^*$ .

Assume now that  $\{h_j(\alpha)\}$  satisfies, in addition to (2.3), the following regularity conditions in a closed subset  $A$  containing  $\alpha^*$  as an interior point:

(H1) There exist constants  $a_j > 0$  such that

$$\sum_{j=0}^{\infty} j a_j < \infty \quad \text{and} \quad |h_j(\alpha)| \leq a_j$$

for all  $j = 0, 1, \dots$  and all  $\alpha \in A$ .

(H2)  $h'_j(\alpha)$  exists and there are constants  $b_j > 0$  such that

$$\sum_{j=0}^{\infty} j b_j < \infty \quad \text{and} \quad |h'_j(\alpha)| \leq b_j$$

for all  $j = 0, 1, \dots$  and all  $\alpha \in A$ .

Under these conditions, if  $C(\alpha^*) < 1$  (i.e, if the signal is captured by  $\mathcal{L}_\alpha$  with  $\alpha = \alpha^*$ ) and the sample size  $n$  is sufficiently large, it has been shown by Li and Kedem (1991) that almost surely the mapping  $\hat{\rho}(\alpha)$  has a unique fixed-point, denoted by  $\hat{\alpha}_n$ , in a neighborhood of  $\alpha^*$ , and the sequence  $\{\hat{\alpha}_n^{(m)}\}$  generated by (2.8) converges monotonically to  $\hat{\alpha}_n$  as  $m \rightarrow \infty$ , provided

the initial estimate  $\hat{\alpha}_n^{(0)}$  is not too far from  $\hat{\alpha}_n$ . Moreover, the estimator  $\hat{\alpha}_n$  converges almost surely to  $\alpha^*$  and thus  $\hat{\omega}_n$  defined by

$$\hat{\omega}_n := \arccos \hat{\alpha}_n = \arccos \hat{\rho}(\hat{\alpha}_n) \quad (2.9)$$

is strongly consistent for estimating  $\omega_0$ . In the following, we refer to both  $\hat{\alpha}_n$  and  $\hat{\omega}_n$  as the *CM estimator*. The main purpose of this paper is to establish the asymptotic normality for  $\hat{\alpha}_n$  and  $\hat{\omega}_n$ .

### 3 ASYMPTOTIC NORMALITY OF SAMPLE AUTOCOVARIANCES

To investigate the asymptotic distribution of the CM estimator, we clearly need the asymptotic distribution of the sample autocovariances defined by (2.7). For this purpose, let us first consider the following general case.

Let  $\{h_j, j = 0, 1, \dots\}$  be the real-valued impulse response sequence of a linear causal filter with  $\sum |h_j| < \infty$ , and let

$$H(\omega) := \sum_{j=0}^{\infty} h_j e^{-ij\omega}$$

be its transfer function. For a given data set  $\{y_0, y_1, \dots, y_{n-1}\}$  obtained from (2.1), define the filtered data by

$$\hat{z}_t := \sum_{j=0}^t h_j y_{t-j} \quad (t = 0, 1, \dots, n-1).$$

We would like to establish the asymptotic normality for the sample autocovariances

$$\hat{r}_j := n^{-1} \sum_{t=0}^{n-j-1} \hat{z}_{t+j} \hat{z}_t \quad (j = 0, 1, \dots, q) \quad (3.1)$$

where  $q$  is a fixed integer with  $0 \leq q < n$ . For convenience, we assume that  $\phi$  is a constant. As will be seen later, this assumption can be eliminated without changing our results.

Note that  $\hat{r}_j$  is defined for a *finite* sample. For simplicity, let us first consider its counterpart  $\tilde{r}_j$  defined on the basis of an *infinite* sample by

$$\tilde{r}_j := n^{-1} \sum_{t=0}^{n-1} z_{t+j} z_t$$

where  $z_t$ , defined by

$$z_t := \sum_{j=0}^{\infty} h_j y_{t-j},$$

requires the entire history of the data  $\{y_t\}$  up to time  $t$ . Using (2.1),  $z_t$  can be rewritten as

$$z_t = \beta_h \cos(\omega_0 t + \phi_h) + \zeta_t \quad (3.2)$$

where

$$\begin{aligned} \beta_h &:= \beta |H(\omega_0)|, & \phi_h &:= \phi + \arg\{H(\omega_0)\}, \\ \zeta_t &:= \sum_{j=0}^{\infty} h_j \epsilon_{t-j} = \sum_j \mu_j \xi_{t-j}, & \text{and } \mu_j &:= \sum_{k=0}^{\infty} h_k \psi_{j-k}. \end{aligned} \quad (3.3)$$

The following lemmas provide some asymptotic properties of  $\{\tilde{r}_j\}$ .

**Lemma 3.1** *Suppose that  $E(\xi_t^4) = \kappa \sigma_\xi^2 < \infty$ . Then*

$$\lim_{n \rightarrow \infty} n \operatorname{cov}(\tilde{r}_j, \tilde{r}_k) = \lim_{n \rightarrow \infty} n E\{(\tilde{r}_j - r_j)(\tilde{r}_k - r_k)\} = v_{jk} < \infty$$

where

$$r_j := \frac{\beta_h^2}{2} \cos(\omega_0 j) + r_j^\zeta, \quad r_j^\zeta := E(\zeta_{t+j} \zeta_t),$$

and

$$\begin{aligned} v_{jk} &:= 2\beta_h^2 \cos(\omega_0 j) \cos(\omega_0 k) \sum_{\tau=-\infty}^{\infty} r_\tau^\zeta \cos(\omega_0 \tau) \\ &\quad + (\kappa - 3) r_j^\zeta r_k^\zeta + \sum_{\tau=-\infty}^{\infty} \left( r_\tau^\zeta r_{\tau+j-k}^\zeta + r_{\tau+j}^\zeta r_{\tau-k}^\zeta \right). \end{aligned} \quad (3.4)$$

PROOF. Using (3.2) and the trigonometric identity

$$\cos \omega \cos \lambda = \frac{\cos(\omega - \lambda) + \cos(\omega + \lambda)}{2}, \quad (3.5)$$

$\tilde{r}_j$  can be written as

$$\tilde{r}_j = r_j + n^{-1} \sum_{t=0}^{n-1} x_{jt} + \beta_h^2 (2n)^{-1} \sum_{t=0}^{n-1} \cos\{\omega_0(2t+j) + 2\phi_h\} \quad (3.6)$$

where

$$x_{jt} := \beta_h \zeta_{t+j} \cos(\omega_0 t + \phi_h) + \beta_h \zeta_t \cos\{\omega_0(t+j) + \phi_h\} + \zeta_{t+j} \zeta_t - r_j^\zeta. \quad (3.7)$$

Note that  $E(x_{jt}) = 0$  and that for any  $j$  and  $\phi_h$ ,

$$\sum_{t=0}^{n-1} \cos\{\omega_0(2t+j) + 2\phi_h\} = O(1).$$

Therefore, (3.6) induces

$$E(\tilde{r}_j) = r_j + O(n^{-1}) \quad \text{and} \quad \tilde{r}_j - r_j = n^{-1} \sum_{t=0}^{n-1} x_{jt} + O(n^{-1}). \quad (3.8)$$

It is readily seen from (3.8) that

$$\begin{aligned} \lim_{n \rightarrow \infty} n \operatorname{cov}(\tilde{r}_j, \tilde{r}_k) &= \lim_{n \rightarrow \infty} n E\{(\tilde{r}_j - r_j)(\tilde{r}_k - r_k)\} \\ &= \lim_{n \rightarrow \infty} n^{-1} \operatorname{cov}\left(\sum_{t=0}^{n-1} x_{jt}, \sum_{t=0}^{n-1} x_{kt}\right). \end{aligned}$$

Moreover, simple algebra shows that

$$n^{-1} \operatorname{cov}\left(\sum_{t=0}^{n-1} x_{jt}, \sum_{t=0}^{n-1} x_{kt}\right) = I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &:= \beta_h^2 n^{-1} \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} \left[ r_{t-s+j-k}^\zeta \cos(\omega_0 t + \phi_h) \cos(\omega_0 s + \phi_h) \right. \\ &\quad + r_{t-s+j}^\zeta \cos(\omega_0 t + \phi_h) \cos\{\omega_0(s+k) + \phi_h\} \\ &\quad + r_{t-s-k}^\zeta \cos\{\omega_0(t+j) + \phi_h\} \cos(\omega_0 s + \phi_h) \\ &\quad \left. + r_{t-s}^\zeta \cos\{\omega_0(t+j) + \phi_h\} \cos\{\omega_0(s+k) + \phi_h\} \right] \\ &:= T_1 + T_2 + T_3 + T_4, \\ I_2 &:= \beta_h n^{-1} \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} [c(t-s+j, k) \cos(\omega_0 t + \phi_h) + c(t-s, k) \cos\{\omega_0(t+j) + \phi_h\} \\ &\quad + c(s-t+k, j) \cos(\omega_0 s + \phi_h) + c(s-t, j) \cos\{\omega_0(s+k) + \phi_h\}], \\ I_3 &:= n^{-1} \operatorname{cov}\left(\sum_{t=0}^{n-1} \zeta_{t+j} \zeta_t, \sum_{t=0}^{n-1} \zeta_{t+k} \zeta_t\right), \end{aligned}$$

and

$$c(j, k) := E(\zeta_{t+j} \zeta_{t+k} \zeta_t)$$

is the third-order cumulant of  $\{\zeta_t\}$ . Using (3.5) and the substitution  $\tau := t - s$ ,  $T_1$  can be written as

$$T_1 = \frac{\beta_h^2}{2} \sum_{|\tau| \leq n-1} r_{\tau+j-k}^\zeta \left[ \left(1 - \frac{|\tau|}{n}\right) \cos(\omega_0 \tau) + \frac{1}{n} \sum_{t \in D} \cos\{\omega_0(2t - \tau) + 2\phi_h\} \right]$$

where  $D := \{t : \max(0, \tau) \leq t \leq \min(n-1, \tau+n-1)\}$ . Since  $\sum |r_\tau^\zeta| < \infty$  and

$$n^{-1} \sum_{t \in D} \cos\{\omega_0(2t - \tau) + 2\phi_h\} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for any  $\tau$ , bounded convergence theorem gives

$$T_1 \rightarrow \frac{\beta_h^2}{2} \sum_{\tau=-\infty}^{\infty} r_{\tau+j-k}^\zeta \cos(\omega_0 \tau) = \frac{\beta_h^2}{2} \sum_{\tau=-\infty}^{\infty} r_\tau^\zeta \cos\{\omega_0(\tau - j + k)\}.$$

Similarly, we obtain

$$\begin{aligned} T_2 &\rightarrow \frac{\beta_h^2}{2} \sum_{\tau=-\infty}^{\infty} r_\tau^\zeta \cos\{\omega_0(\tau - j - k)\} \\ T_3 &\rightarrow \frac{\beta_h^2}{2} \sum_{\tau=-\infty}^{\infty} r_\tau^\zeta \cos\{\omega_0(\tau + j + k)\} \\ T_4 &\rightarrow \frac{\beta_h^2}{2} \sum_{\tau=-\infty}^{\infty} r_\tau^\zeta \cos\{\omega_0(\tau + j - k)\}. \end{aligned}$$

Combining these together yields

$$I_1 \rightarrow 2\beta_h^2 \cos(\omega_0 j) \cos(\omega_0 k) \sum_{\tau=-\infty}^{\infty} r_\tau^\zeta \cos(\omega_0 \tau).$$

In addition, since  $\sum |c(\tau + j, k)| < \infty$  for any fixed  $j$  and  $k$ , an analogous argument leads to  $I_2 \rightarrow 0$ . Finally, Proposition 7.3.1 (Brockwell and Davis, 1987) gives

$$I_3 \rightarrow (\kappa - 3)r_j^\zeta r_k^\zeta + \sum_{\tau=-\infty}^{\infty} \left( r_\tau^\zeta r_{\tau+j-k}^\zeta + r_{\tau+j}^\zeta r_{\tau-k}^\zeta \right).$$

The assertion is thus proved upon combining these results.  $\diamond$

**Lemma 3.2** *Under the same assumption as Lemma 3.1,  $\{n^{1/2}(\tilde{r}_j - r_j), j = 0, \dots, q\}$  are asymptotically jointly normal with mean zero and covariance matrix  $\mathbf{V} := (v_{jk})_{j,k=0}^q$ , where  $v_{jk}$  is defined by (3.4).*

PROOF. Using the Cramér-Wold device (Brockwell and Davis, 1987, Proposition 6.3.1), we only need to show that for any  $\{\lambda_j, j = 0, \dots, q\} \neq \{0\}$ ,

$$n^{1/2} \sum_{j=0}^q \lambda_j (\tilde{r}_j - r_j) \rightarrow N(0, v) \quad \text{in distribution,}$$

where  $v := \sum \sum \lambda_j \lambda_k v_{jk} > 0$ . From (3.8), it suffices to show that

$$n^{-1/2} \sum_{t=0}^{n-1} x_t \rightarrow N(0, v) \quad \text{in distribution,} \quad (3.9)$$

where  $x_t := \sum \lambda_j x_{jt}$ , and  $x_{jt}$  is defined by (3.7). If the sequence  $\{\mu_j\}$  given by (3.3) has a finite length, i.e., if  $\mu_j \equiv 0$  for all  $|j| > m$  with  $m$  being a positive integer, then, after some straightforward manipulations similar to the proof of Theorem 6.7 (Hall and Heyde, 1980, pp. 192), we can write

$$n^{-1/2} \sum_{t=0}^{n-1} x_t = n^{-1/2} \sum_{t=1}^n M_t + o_p(1)$$

where

$$\begin{aligned} M_t &:= B_t \xi_t + \sum_{\tau=0}^{2m+q} A_\tau (\xi_t \xi_{t-\tau} - \sigma_\xi^2 \delta_\tau), \quad A_\tau := \sum_{j=0}^q \lambda_j A_{j\tau}, \\ B_t &:= \sum_{j=0}^q \lambda_j \sum_k \mu_k \beta_h [\cos\{\omega_0(t+k+j) + \phi_h\} + \cos\{\omega_0(t+k-j) + \phi_h\}], \\ A_{j0} &:= r_j^\zeta / \sigma_\xi^2, \quad \text{and} \quad A_{j\tau} := (r_{j+\tau}^\zeta + r_{j-\tau}^\zeta) / \sigma_\xi^2, \quad \text{for } \tau \neq 0. \end{aligned}$$

It is easy to see that  $\{M_t\}$  are martingale differences with respect to the  $\sigma$ -fields  $\mathcal{F}_t$  generated by  $\{\xi_j, j \leq t\}$  for  $t \geq 0$ . Direct computations, followed by an application of the law of large numbers, yield

$$n^{-1} V_n^2 := n^{-1} \sum_{t=1}^n E(M_t^2 | \mathcal{F}_{t-1}) \rightarrow v \quad \text{in probability.}$$

It can also be shown by direct computations that

$$n^{-1} s_n^2 := n^{-1} E(V_n^2) \rightarrow v. \quad (3.10)$$

Therefore,  $V_n^2 / s_n^2 \rightarrow 1$  in probability. Furthermore, (3.10) implies that the Lindeberg condition (Brown, 1971, eq. 2) is equivalent to

$$n^{-1} \sum_{t=1}^n E\{M_t^2 I(|M_t| \geq \varepsilon s_t)\} \rightarrow 0,$$

where  $I(\cdot)$  is the indicator function. Hence it suffices to verify that

$$E\{M_t^2 I(|M_t| \geq \varepsilon s_t)\} \rightarrow 0$$

for any  $\varepsilon > 0$ . To do so, we note that  $|B_t| \leq B$  for some  $B > 0$  and all  $t$ . Therefore,

$$|M_t| \leq U_t := B|\xi_t| + \left| \sum_{\tau=0}^{2m+q} A_\tau(\xi_t \xi_{t-\tau} - \sigma_\xi^2 \delta_\tau) \right|.$$

Moreover, (3.10) implies that  $s_t \geq \varepsilon t^{1/2}$  for small  $\varepsilon$  and large  $t$ . Combining these results gives

$$E\{M_t^2 I(|M_t| \geq \varepsilon s_t)\} \leq E\{U_t^2 I(U_t \geq \varepsilon^2 t^{1/2})\} = E\{U_0^2 I(U_0 \geq \varepsilon^2 t^{1/2})\} \rightarrow 0,$$

where the second expression is due to the stationarity of  $\{U_t\}$ . Applying Theorem 2 (Brown, 1971) proves (3.9).

If  $\{\mu_j\}$  is of infinite length, (3.9) can also be verified by following the proof of Proposition 7.3.3 (Brockwell and Davis, 1987). In fact, for any  $m > 0$ , let us define

$$\begin{aligned} \zeta_t^m &:= \sum_{j=-m}^m \mu_j \xi_{t-j}, & z_t^m &:= \beta_h \cos(\omega_0 t + \phi_h) + \zeta_t^m, \\ \tilde{r}_j^m &:= n^{-1} \sum_{t=0}^{n-1} z_{t+j}^m z_t^m, & \text{and } r_j^m &:= \frac{\beta_h^2}{2} \cos(\omega_0 j) + E(\zeta_{t+j}^m \zeta_t^m). \end{aligned}$$

If so doing, we obtain, for any fixed  $m$ ,

$$n^{1/2} \sum_{j=0}^q \lambda_j (\tilde{r}_j^m - r_j^m) = S_n^m + O(n^{-1/2})$$

and

$$S_n^m := n^{-1/2} \sum_{t=0}^{n-1} x_t^m \rightarrow S^m \sim N(0, v^m)$$

in distribution as  $n \rightarrow \infty$ , where  $x_t^m := \sum \lambda_j x_{jt}^m$  and  $x_{jt}^m$  is given by (3.7) with  $\zeta_t^m$  in place of  $\zeta_t$ , and where  $v^m := \sum \sum \lambda_j \lambda_k v_{jk}^m$  and  $v_{jk}^m$  is defined by (3.4) with autocovariances of  $\{\zeta_t\}$  replaced by autocovariances of  $\{\zeta_t^m\}$ . Since  $v^m \rightarrow v$  as  $m \rightarrow \infty$ , then  $S^m \rightarrow N(0, v)$  in distribution. Moreover, an analogous argument as in the proof of Proposition 7.3.3 leads to

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{pr}(n^{1/2} |\tilde{r}_j^m - r_j^m - \tilde{r}_j + r_j| > \varepsilon) = 0$$

for any  $\varepsilon > 0$ . The proof is then completed by applying Proposition 6.3.9 (Brockwell and Davis, 1987).  $\diamond$



**REMARK 3.1** The asymptotic normality has been proved by Mackisack and Poskitt (1989) for the simplest case where  $\{\zeta_t\}$  are independent and identically distributed (see also Stoica *et al.*, 1989). Therefore, Lemma 3.2 is an extension of their results to the case of *colored* noise.  $\diamond$

With the help of the above lemmas, we are now able to state the central limit theorem for the sample autocovariances  $\{\hat{r}_j\}$  given by (3.1).

**Theorem 3.1** *Suppose that  $E(\xi_t^4) = \kappa\sigma_\xi^2 < \infty$  and that*

$$\sum_{j=0}^{\infty} j|h_j| < \infty.$$

*Then,  $\{n^{1/2}(\hat{r}_j - r_j), j = 0, \dots, q\}$  are asymptotically jointly normal with mean zero and covariance matrix  $\mathbf{V}$ .*

**PROOF.** By Lemma 3.2, it suffices to show that  $n^{1/2}(\tilde{r}_\tau - \hat{r}_\tau) \rightarrow 0$  in probability as  $n \rightarrow \infty$ . To do so, we first claim that

$$n^{1/2}(\tilde{r}_\tau - \check{r}_\tau) \rightarrow 0 \quad \text{in probability,} \quad (3.11)$$

where

$$\check{r}_\tau := n^{-1} \sum_{t=0}^{n-1} \hat{z}_{t+\tau} \hat{z}_t.$$

In fact, simple algebra gives  $z_t = \hat{z}_t + \tilde{z}_t$  and

$$\begin{aligned} n^{1/2}(\tilde{r}_\tau - \check{r}_\tau) &= n^{-1/2} \sum_{t=0}^{n-1} (\hat{z}_{t+\tau} \tilde{z}_t + \tilde{z}_{t+\tau} \hat{z}_t + \tilde{z}_{t+\tau} \tilde{z}_t) \\ &:= I_1 + I_2 + I_3 \end{aligned}$$

where

$$\tilde{z}_t := \sum_{j=t+1}^{\infty} h_j y_{t-j}.$$

Using (2.1),  $I_1$  can be written as

$$\begin{aligned} I_1 &= n^{-1/2} \sum_{t=0}^{n-1} \sum_{j=t+1}^{\infty} \sum_{k=0}^{t+\tau} h_j h_k [\beta^2 \cos\{\omega_0(t-j) + \phi\} \cos\{\omega_0(t+\tau-k) + \phi\} \\ &\quad + \beta \epsilon_{t-j} \cos\{\omega_0(t+\tau-k) + \phi\} \\ &\quad + \beta \epsilon_{t+\tau-k} \cos\{\omega_0(t-j) + \phi\} \\ &\quad + \epsilon_{t-j} \epsilon_{t+\tau-k}] \\ &:= T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Clearly, we have

$$|T_1| \leq n^{-1/2} \beta^2 H \sum_{t=0}^{n-1} \sum_{j=t+1}^{\infty} |h_j| \leq n^{-1/2} \beta^2 H \sum_{j=0}^{\infty} j |h_j| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where  $H := \sum |h_j|$ . Similarly,

$$E(|T_2|) \leq n^{-1/2} E(|\epsilon_0|) \beta H \sum_{j=0}^{\infty} j |h_j| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and thus  $T_2 \rightarrow 0$  in probability by applying Chebyshev's inequality. In the same way, we can show that  $T_3$  and  $T_4$  vanish in probability as well. Combining these results gives  $I_1 \rightarrow 0$  in probability. The same conclusion can also be made about  $I_2$  and  $I_3$  by an analogous argument. (3.11) is thus proved. Finally, we claim that  $n^{1/2}(\tilde{r}_\tau - \hat{r}_\tau) \rightarrow 0$  in probability. This can be easily established upon noting that

$$n^{1/2} E(|\tilde{r}_\tau - \hat{r}_\tau|) \leq n^{-1/2} \sum_{t=n-\tau}^{n-1} \sum_{j=0}^t \sum_{k=0}^{t+\tau} |h_j h_k| E(|y_{t+\tau-k} y_{t-j}|) \leq n^{-1/2} \tau K H^2$$

where the last inequality is due to the fact that

$$E(|y_t y_s|) \leq K := \beta^2 + 2\beta E(|\epsilon_0|) + E(\epsilon_0^2).$$

The theorem is thus proved.  $\diamond$

**REMARK 3.2** Theorem 3.1 remains valid if  $\phi$  is random and independent of  $\{\xi_t\}$ . In fact, by conditioning on  $\phi$ , Theorem 3.1 can be proved in the same way. Since the asymptotic *conditional* distribution given  $\phi$  does not depend on  $\phi$ , the same result holds for the asymptotic *unconditional* distribution as well.  $\diamond$

**REMARK 3.3** This theorem can be easily generalized to the case of multiple sinusoids in additive noise. In fact, suppose that  $\{y_t\}$  is given by

$$y_t := \sum_{\nu=0}^{q-1} \beta_\nu \cos(\omega_\nu t + \phi_\nu) + \epsilon_t,$$

where  $q$  is a positive integer,  $\beta_\nu$  and  $\omega_\nu$  are constants with  $\beta_\nu > 0$  and  $0 < \omega_1 < \dots < \omega_q < \pi$ , and  $\phi_\nu \in (-\pi, \pi]$  can be either constants or random and independent of the noise  $\{\epsilon_t\}$  given by (2.2). Then, Theorem 3.1 still holds with  $v_{jk}$  defined by

$$\begin{aligned} v_{jk} &:= 2 \sum_{\nu=0}^{q-1} \left\{ \beta_{\nu h}^2 \cos(\omega_\nu j) \cos(\omega_\nu k) \sum_{\tau=-\infty}^{\infty} r_\tau^\zeta \cos(\omega_\nu \tau) \right\} \\ &\quad + (\kappa - 3) r_j^\zeta r_k^\zeta + \sum_{\tau=-\infty}^{\infty} \left( r_\tau^\zeta r_{\tau+j-k}^\zeta + r_{\tau+j}^\zeta r_{\tau-k}^\zeta \right) \end{aligned}$$

where  $\beta_{\nu h} := \beta |H(\omega_\nu)|$ .  $\diamond$

By a somewhat different approach, Yakowitz (1991b) have given a result similar to Theorem 3.1. But in his work, only a lower bound of the asymptotic variance is provided and the result is restricted to FIR filters.

As a result of Theorem 3.1, we can also establish the asymptotic normality for the sample autocorrelation

$$\hat{\rho}_j := \hat{r}_j / \hat{r}_0$$

as follows.

**Corollary 3.1** *Suppose that conditions in Theorem 3.1 are satisfied. Then,  $n^{1/2}(\hat{\rho}_j - \rho_j)$  is asymptotically normal with mean zero and variance  $v_j$ , where  $\rho_j := r_j / r_0$  and*

$$v_j := (\rho_j^2 v_{00} - 2\rho_j v_{0j} + v_{jj}) / r_0^2. \quad (3.12)$$

PROOF. The assertion follows from Theorem 3.1 by applying Proposition 6.4.3 (Brockwell and Davis, 1987).  $\diamond$

## 4 ASYMPTOTIC NORMALITY OF THE CM ESTIMATOR

Applying the general results obtained in the proceeding section, we would like in this section to establish the asymptotic normality for the CM estimators  $\hat{\alpha}_n$  and  $\hat{\omega}_n$  defined in Section 2. For this purpose, we first have

**Lemma 4.1** *Let  $\{y_t\}$  be defined by (2.1)-(2.2) and assume that  $E(\xi_t^4) < \infty$ . Suppose in addition that*

$$\sum_{j=0}^{\infty} j |h_j(\alpha^*)| < \infty.$$

*Then,  $n^{1/2}\{\hat{\rho}(\alpha^*) - \rho(\alpha^*)\}$  is asymptotically normal with mean zero and variance  $\sigma_\rho^2$ , where  $\hat{\rho}(\alpha^*)$  is given by (2.6) with  $\alpha = \alpha^*$ , and  $\sigma_\rho^2 := v_\rho C^2(\alpha^*)$  with*

$$v_\rho := \sum_{k=-\infty}^{\infty} \left\{ (1 + 2\alpha^{*2})(\rho_k^\epsilon(\alpha^*))^2 - 4\alpha^* \rho_k^\epsilon(\alpha^*) \rho_{k+1}^\epsilon(\alpha^*) + \rho_{k+1}^\epsilon(\alpha) \rho_{k-1}^\epsilon(\alpha) \right\}. \quad (4.1)$$

PROOF. Applying Corollary 3.1 with  $h_j := h_j(\alpha^*)$  and  $\zeta_t := \epsilon_t(\alpha^*)$ , we are able to obtain

$$n^{1/2}\{\hat{\rho}(\alpha^*) - \rho(\alpha^*)\} \rightarrow N(0, v_1) \quad \text{in distribution,}$$

where  $v_1$  is defined by (3.12) with  $j = 1$ . Note that in (3.12),  $r_0 = \sigma_\zeta^2/C(\alpha^*)$  where  $\sigma_\zeta^2 := E(\zeta_t^2) = r_0^\zeta$ . Note also that  $\rho(\alpha^*) = \rho_1^\epsilon(\alpha^*) = \alpha^* = \cos \omega_0$ . Therefore, (3.4) gives

$$\begin{aligned} v_{00}/\sigma_\zeta^4 &= S + 2 \sum_{k=-\infty}^{\infty} (\rho_k^\epsilon(\alpha^*))^2 \\ v_{01}/\sigma_\zeta^4 &= \alpha^* S + 2 \sum_{k=-\infty}^{\infty} \rho_k^\epsilon(\alpha^*) \rho_{k+1}^\epsilon(\alpha^*) \\ v_{11}/\sigma_\zeta^4 &= \alpha^{*2} S + \sum_{k=-\infty}^{\infty} \left\{ (\rho_k^\epsilon(\alpha^*))^2 + \rho_{k+1}^\epsilon(\alpha^*) \rho_{k-1}^\epsilon(\alpha^*) \right\} \end{aligned}$$

where  $S := (2\beta_h^2/\sigma_\zeta^2) \sum \rho_k^\epsilon(\alpha^*) \cos(\omega_0 k) + \kappa - 3$ . Simple algebra yields  $v_1 = \sigma_\rho^2$ .  $\diamond$

REMARK 4.1 Because of asymptotic equivalence, there are many other ways of estimating  $\rho(\alpha)$  without changing the results in Lemma 4.1. In fact,  $\hat{\rho}(\alpha)$  can be taken, for example, as the least squares estimator that minimizes

$$\sum_{t=1}^{n-1} \{\hat{y}_t(\alpha) - \rho \hat{y}_{t-1}(\alpha)\}^2 + \sum_{t=1}^{n-1} \{\hat{y}_{t-2}(\alpha) - \rho \hat{y}_{t-1}(\alpha)\}^2.$$

In this case,

$$\hat{\rho}(\alpha) = \frac{\sum_{t=1}^{n-1} \hat{y}_{t-1}(\alpha) \{\hat{y}_t(\alpha) + \hat{y}_{t-2}(\alpha)\}}{2 \sum_{t=1}^{n-1} \hat{y}_{t-1}^2(\alpha)}. \quad (4.2)$$

Some other estimators of  $\rho(\alpha)$  can be found in Li and Kedem (1991).  $\diamond$

Using this lemma, we are able to obtain the following central limit theorem for the CM estimator:

**Theorem 4.1** *Let  $\{y_t\}$  be defined by (2.1)-(2.2) and suppose that  $E(\xi_t^4) < \infty$ . Assume further that (H1), (H2) are satisfied and that  $G(\alpha) \geq g > 0$  for all  $\alpha$  in a neighborhood of  $\alpha^*$ . Then,  $n^{1/2}(\hat{\alpha}_n - \alpha^*)$  and  $n^{1/2}(\hat{\omega}_n - \omega_0)$  are asymptotically normal with mean zero and variances*

$$\sigma_\alpha^2 = \frac{v_\rho}{\gamma^2(\alpha^*)} \quad \text{and} \quad \sigma_\omega^2 = \frac{v_\rho}{\gamma^2(\cos \omega_0) \sin^2 \omega_0}, \quad (4.3)$$

respectively.

PROOF. Let  $e_n(\alpha) := \hat{\rho}(\alpha) - \rho(\alpha)$ . Since  $\hat{\alpha}_n$  is a fixed-point of  $\hat{\rho}(\alpha)$ , then  $\hat{\rho}(\hat{\alpha}_n) = \hat{\alpha}_n$ . Combining this with (2.5) yields

$$e_n(\hat{\alpha}_n) = G(\hat{\alpha}_n)(\hat{\alpha}_n - \alpha^*).$$

Under assumption (H2),  $e_n(\alpha)$  is differentiable with probability one. Therefore, we have the Taylor expansion  $G(\hat{\alpha}_n)(\hat{\alpha}_n - \alpha^*) = e_n(\alpha^*) + e'_n(\tilde{\alpha}_n)(\hat{\alpha}_n - \alpha^*)$ , or,

$$\{G(\hat{\alpha}_n) - e'_n(\tilde{\alpha}_n)\}(\hat{\alpha}_n - \alpha^*) = e_n(\alpha^*), \quad (4.4)$$

where  $\tilde{\alpha}_n$  lies between  $\alpha^*$  and  $\hat{\alpha}_n$ . According to Theorem 5.1 and Corollary 5.1 (Li and Kedem, 1991), we have, almost surely,  $\hat{\alpha}_n \rightarrow \alpha^*$ , and  $e'_n(\alpha) = \hat{\rho}'(\alpha) - \rho'(\alpha) \rightarrow 0$  uniformly in a neighborhood of  $\alpha^*$ . Therefore, by the continuity of  $G(\alpha)$ , we obtain

$$G(\hat{\alpha}_n) - e'_n(\tilde{\alpha}_n) \rightarrow G(\alpha^*) \quad \text{in probability.}$$

Moreover, Lemma 4.1 guarantees that  $n^{1/2}e_n(\alpha^*)$  is asymptotically normal with mean zero and variance  $\sigma_\rho^2$ . The asymptotic normality of  $\hat{\alpha}_n$  with  $\sigma_\alpha^2 := \sigma_\rho^2/G^2(\alpha^*)$  is thus proved by combining all these results and applying Slutsky's theorem to (4.4). From (4.1) and the definitions of  $G(\alpha)$  and  $C(\alpha)$ , we obtain  $\sigma_\alpha^2 = v_\rho/\gamma^2(\alpha^*)$ . Finally, since  $\hat{\omega}_n = \arccos \hat{\alpha}_n$  and

$$\left. \frac{d \arccos \alpha}{d\alpha} \right|_{\alpha=\alpha^*} = \frac{-1}{|\sin \omega_0|},$$

the asymptotic normality of  $\hat{\omega}_n$  follows from Proposition 6.4.1 (Brockwell and Davis, 1987).  $\diamond$

REMARK 4.2 The formula (4.3) shows that in order to reduce the asymptotic variances, the filter must be chosen so as to enhance the sinusoidal signal when  $\alpha$  takes on the true value  $\alpha^*$ .  $\diamond$

## 5 THE AR(2) FILTER

Consider the following AR(2) filter defined recursively by

$$y_t(\alpha) + \theta(\alpha)\eta y_{t-1}(\alpha) + \eta^2 y_{t-2}(\alpha) = y_t \quad (t = 0, 1, \dots, n-1), \quad (5.1)$$

where  $0 < \eta < 1$ , and

$$\theta(\alpha) := -\frac{1 + \eta^2}{\eta} \alpha. \quad (5.2)$$

For any  $|\alpha| \leq 2\eta/(1 + \eta^2)$ , since  $|\theta(\alpha)| \leq 2$ , we can always write

$$\theta(\alpha) = -2 \cos\{\omega(\alpha)\} \quad (5.3)$$

for some  $\omega(\alpha) \in [0, \pi]$ . It turns out that  $\eta \exp\{\pm i\omega(\alpha)\}$  are poles of the AR(2) filter defined by (5.1). Moreover, it can be verified (Li and Kedem, 1991) that the fundamental property (2.3) is satisfied by this filter whenever  $\{\epsilon_t\}$  is white noise, i.e., whenever  $\epsilon_t$  are independent and identically distributed with mean zero and variance  $\sigma_\epsilon^2$ . Therefore, in this case, the filter (5.1) can be applied to estimate the frequency  $\omega_0$  by using the FPI procedure (2.8) and by taking

$$\hat{\omega}_n^{(m)} := \arccos \hat{\rho}(\hat{\alpha}_n^{(m-1)}) = \arccos \hat{\alpha}_n^{(m)} \quad (m = 1, 2, \dots). \quad (5.4)$$

Here  $\hat{\rho}(\alpha)$  can be defined by either (2.6) or (4.2), and  $\hat{y}_t(\alpha)$  is given by (5.1) with zero initial values. This procedure is clearly an iterative filtering algorithm. In fact, at the  $m$ th iteration, we first filter the sample  $\{y_t\}$  by (5.1) with  $\alpha = \hat{\alpha}_n^{(m-1)}$ , and then calculate  $\hat{\rho}(\hat{\alpha}_n^{(m-1)})$  on the basis of the output using (2.6) or (4.2). Finally, we obtain  $\hat{\omega}_n^{(m)}$  by (5.4) as an estimate of  $\omega_0$  and  $\hat{\alpha}_n^{(m)}$  by (2.8) for the next iteration.

It has also been verified (Li and Kedem, 1991) that the filter (5.1) satisfies (H1) and (H2) with  $C(\alpha^*) < 1$ . Therefore, it is guaranteed with probability one for large  $n$  that the mapping  $\hat{\rho}(\alpha)$  has a unique fixed-point  $\hat{\alpha}_n$  in the vicinity of  $\alpha^*$ , and that  $\hat{\omega}_n^{(m)}$  defined by (5.4) converges monotonically to  $\hat{\omega}_n = \arccos \hat{\alpha}_n$  as  $m \rightarrow \infty$ . Furthermore,  $\hat{\omega}_n$  is strongly consistent for estimating  $\omega_0$  and, according to Theorem 4.1,  $n^{1/2}(\hat{\omega}_n - \omega_0)$  is asymptotically normal with mean zero and variance  $\sigma_\omega^2$  defined by (4.3).

To obtain explicit expressions for  $\sigma_\alpha^2$  and  $\sigma_\omega^2$ , we first note that for the AR(2) filter (5.1),

$$|H(\omega_0; \alpha^*)|^2 = \frac{1}{(1 - \eta^2)^2 \sin^2 \omega_0}.$$

Using the formulas given by He and Kedem (1990b) for AR(2) filters, we obtain

$$E\{\epsilon_t^2(\alpha)\} = \frac{\sigma_\epsilon^2}{\sin^2\{\omega(\alpha)\}} \sum_{j=0}^{\infty} \sin^2\{\omega(\alpha)(j+1)\} \eta^{2j},$$

and

$$\rho_k^\epsilon(\alpha) = \cos\{\omega(\alpha)k\} \eta^k + \frac{1 - \eta^2}{1 + \eta^2} \cot\{\omega(\alpha)\} \sin\{\omega(\alpha)k\} \eta^k \quad (5.5)$$

for  $k \geq 0$ , where  $\omega(\alpha)$  is defined by (5.3). It follows by straightforward computations that

$$E\{\epsilon_t^2(\alpha)\} = \frac{\sigma_\epsilon^2}{(1 - \eta^4)(1 - \alpha^2)},$$

and, therefore, we can write

$$\gamma(\alpha^*) = \frac{1 + \eta^2}{1 - \eta^2} \gamma$$

where  $\gamma := \beta^2/(2\sigma_\epsilon^2)$  is the signal-to-noise ratio (SNR) of  $\{y_t\}$ . It is readily seen that  $\gamma(\alpha^*)$ , the signal-to-noise ratio of the filtered data at  $\alpha^*$ , does not depend on  $\alpha^*$ . This means that the sinusoidal signal is equally enhanced by the AR(2) filter (5.1) for any  $\omega_0 \in (0, \pi)$  satisfying  $|\cos \omega_0| < 2\eta/(1 + \eta^2)$ .

In addition, direct but lengthy calculation of (4.1) using (5.5) yields

$$v_\rho = \frac{1 - \eta^2}{1 + \eta^2} (1 - \alpha^{*2}).$$

Therefore, from (4.3), we obtain

$$\sigma_\alpha^2 = \left( \frac{1 - \eta^2}{1 + \eta^2} \right)^3 \frac{1 - \alpha^{*2}}{\gamma^2} \quad \text{and} \quad \sigma_\omega^2 = \left( \frac{1 - \eta^2}{1 + \eta^2} \right)^3 \frac{1}{\gamma^2}. \quad (5.6)$$

Clearly, in order to achieve a small asymptotic variance,  $\eta$  should be chosen as close to 1 as possible.

Given (5.6), a natural question is what we can learn from these formulas about the mean-square error (MSE)  $E(\hat{\omega}_n - \omega_0)^2$  of the CM estimator  $\hat{\omega}_n$ . Since a closed-form expression of the MSE is difficult, if not impossible, to obtain, we would like to investigate this problem based on a numerical experiment. First of all, it should be noted from (5.6) that  $\sigma_\omega^2$  tends to zero as  $\eta \rightarrow 1$ . Therefore, it is reasonable to believe that in order for the asymptotic variance  $\sigma_\omega^2/n$  of  $\hat{\omega}_n$  to be a good approximation of the MSE, a larger sample size  $n$  is required when  $\eta$  becomes closer to 1, as vindicated by the simulation results shown in Table 1.

For various values of  $\eta$  and  $n$ , Table 1 presents the estimated MSE of the normalized CM estimator  $\hat{\omega}_n/\pi$  on the basis of 100 independent realizations. Each realization was generated from the model (2.1)-(2.2) with  $\{\epsilon_t\}$  being Gaussian white noise, and with  $\omega_0 = 0.42\pi$ ,  $\phi = 0.1\pi$ , and  $\gamma = 1$  (i.e., SNR = 0 dB). Figure 1 gives the corresponding curves obtained from Table 1. To produce the CM estimates, the FPI procedure (2.8) was used in connection with  $\hat{\rho}(\alpha)$  given

Table 1: MSE of CM Estimator by FPI with AR(2) Filter (SNR=0dB)

$n$	$\eta$						Asymptotic
	0.85	0.90	0.95	0.98	0.99	0.999	Var of NLS
100	$7.25 \times 10^{-6}$	$3.06 \times 10^{-6}$	$1.53 \times 10^{-6}$	$1.31 \times 10^{-6}$	$1.28 \times 10^{-6}$	$1.24 \times 10^{-6}$	$1.22 \times 10^{-6}$
	$4.24 \times 10^{-6}$	$1.17 \times 10^{-6}$	$1.36 \times 10^{-7}$	$8.35 \times 10^{-9}$	$1.03 \times 10^{-9}$	$1.01 \times 10^{-12}$	
300	$1.71 \times 10^{-6}$	$5.35 \times 10^{-7}$	$1.21 \times 10^{-7}$	$5.55 \times 10^{-8}$	$5.13 \times 10^{-8}$	$5.01 \times 10^{-8}$	$4.50 \times 10^{-8}$
	$1.41 \times 10^{-6}$	$3.91 \times 10^{-7}$	$4.55 \times 10^{-8}$	$2.78 \times 10^{-9}$	$3.43 \times 10^{-10}$	$3.38 \times 10^{-13}$	
500	$8.97 \times 10^{-7}$	$2.82 \times 10^{-7}$	$5.22 \times 10^{-8}$	$1.44 \times 10^{-8}$	$1.10 \times 10^{-8}$	$1.04 \times 10^{-8}$	$9.73 \times 10^{-9}$
	$8.47 \times 10^{-7}$	$2.34 \times 10^{-7}$	$2.73 \times 10^{-8}$	$1.67 \times 10^{-9}$	$2.06 \times 10^{-10}$	$2.03 \times 10^{-13}$	
700	$8.29 \times 10^{-7}$	$2.47 \times 10^{-7}$	$3.91 \times 10^{-8}$	$7.14 \times 10^{-9}$	$3.94 \times 10^{-9}$	$3.28 \times 10^{-9}$	$3.54 \times 10^{-9}$
	$6.05 \times 10^{-7}$	$1.67 \times 10^{-7}$	$1.95 \times 10^{-8}$	$1.19 \times 10^{-9}$	$1.47 \times 10^{-10}$	$1.45 \times 10^{-13}$	
900	$5.43 \times 10^{-7}$	$1.67 \times 10^{-7}$	$2.81 \times 10^{-8}$	$5.58 \times 10^{-9}$	$2.64 \times 10^{-9}$	$1.64 \times 10^{-9}$	$1.67 \times 10^{-9}$
	$4.71 \times 10^{-7}$	$1.30 \times 10^{-7}$	$1.52 \times 10^{-8}$	$9.28 \times 10^{-10}$	$1.14 \times 10^{-10}$	$1.13 \times 10^{-13}$	
1100	$3.58 \times 10^{-7}$	$1.05 \times 10^{-7}$	$1.75 \times 10^{-8}$	$3.46 \times 10^{-9}$	$1.74 \times 10^{-9}$	$1.15 \times 10^{-9}$	$9.13 \times 10^{-10}$
	$3.85 \times 10^{-7}$	$1.07 \times 10^{-7}$	$1.24 \times 10^{-8}$	$7.59 \times 10^{-10}$	$9.35 \times 10^{-11}$	$9.22 \times 10^{-14}$	
1300	$3.49 \times 10^{-7}$	$9.66 \times 10^{-8}$	$1.36 \times 10^{-8}$	$2.08 \times 10^{-9}$	$9.09 \times 10^{-10}$	$6.13 \times 10^{-10}$	$5.53 \times 10^{-10}$
	$3.26 \times 10^{-7}$	$9.02 \times 10^{-8}$	$1.05 \times 10^{-8}$	$6.42 \times 10^{-10}$	$7.91 \times 10^{-11}$	$7.81 \times 10^{-14}$	
1500	$2.31 \times 10^{-7}$	$7.54 \times 10^{-8}$	$1.39 \times 10^{-8}$	$2.25 \times 10^{-9}$	$8.08 \times 10^{-10}$	$3.85 \times 10^{-10}$	$3.60 \times 10^{-10}$
	$2.82 \times 10^{-7}$	$7.81 \times 10^{-8}$	$9.09 \times 10^{-9}$	$5.57 \times 10^{-10}$	$6.86 \times 10^{-11}$	$6.76 \times 10^{-14}$	
1700	$2.33 \times 10^{-7}$	$6.83 \times 10^{-8}$	$1.00 \times 10^{-8}$	$1.35 \times 10^{-9}$	$5.55 \times 10^{-10}$	$2.82 \times 10^{-10}$	$2.47 \times 10^{-10}$
	$2.49 \times 10^{-7}$	$6.89 \times 10^{-8}$	$8.02 \times 10^{-9}$	$4.91 \times 10^{-10}$	$6.05 \times 10^{-11}$	$5.97 \times 10^{-14}$	
1900	$2.21 \times 10^{-7}$	$6.41 \times 10^{-8}$	$9.17 \times 10^{-9}$	$1.34 \times 10^{-9}$	$5.12 \times 10^{-10}$	$2.16 \times 10^{-10}$	$1.77 \times 10^{-10}$
	$2.23 \times 10^{-7}$	$6.17 \times 10^{-8}$	$7.18 \times 10^{-9}$	$4.40 \times 10^{-10}$	$5.41 \times 10^{-11}$	$5.34 \times 10^{-14}$	



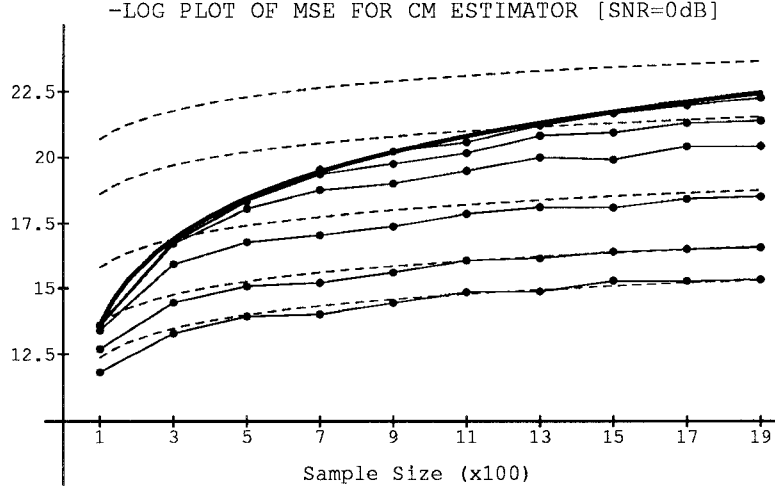


Figure 1:  $-\log$  plot of MSE of the CM estimator for (from the bottom)  $\eta=0.85, 0.90, 0.95, 0.98, 0.99$ , and  $0.999$ . The dashed lines (---) are asymptotic variances of the CM estimator for (from the bottom)  $\eta=0.85, 0.90, 0.95, 0.98$ , and  $0.99$ . The solid line is the asymptotic variance of the NLS.

by (4.2). For  $\eta = 0.85$ , the initial value was taken to be  $\hat{\alpha}_n^{(0)} = \cos(0.6\pi)$  (i.e.,  $\hat{\omega}_n^{(0)} = 0.6\pi$ ). The FPI was terminated according to the stopping criterion:  $m \leq 20$  or  $|\hat{\omega}_n^{(m)} - \hat{\omega}_n^{(m-1)}| < \pi \times 10^{-5}$ , and the resulting estimate was used to initiate the FPI for the next (larger) value of  $\eta$ . The corresponding values of  $\sigma_\omega^2/(\pi^2 n)$  are shown in every second line.

Two conclusions can be drawn immediately from these results. (a) Given a fixed  $\eta$ , the closer it is to 1, the larger the sample size  $n$  is required for  $\sigma_\omega^2/n$  to be a good approximation of  $E(\hat{\omega}_n - \omega_0)^2$ . For instance, if  $\eta = 0.85$ , good approximations are given for  $n \geq 300$ ; while for  $\eta = 0.95$ , we need  $n \geq 1700$ . (b) Given a fixed sample size  $n$ , the MSE can be reduced by increasing  $\eta$ . In the limiting case as  $\eta \rightarrow 1$ , the CM estimator achieves the same efficiency as does the nonlinear least squares (NLS) estimator. Indeed, as shown by Quinn and Fernandes (1988), in the case of  $\eta = 1$  we have

$$n^{3/2}(\hat{\omega}_n - \omega_0) \rightarrow N(0, \sigma_{NLS}^2) \quad \text{in distribution,}$$

where  $\sigma_{NLS}^2 := 12/\gamma$ . Therefore, for fixed  $n$ , the asymptotic variance  $\sigma_{NLS}^2/n^3$  of the NLS estimator can be utilized as an approximation of the MSE, if  $\eta$  is very close to 1.

Finally, it should be pointed out that the Quinn-Fernandes procedure (a special case of the CM method using the FPI and the AR(2) filter with  $\hat{\rho}(\alpha)$  given by (4.2) and  $\eta = 1$ ) requires the initial estimate to be of accuracy  $o(n^{-1/2})$  (see Quinn and Fernandes, 1988), and therefore they suggested using some other methods to initiate the procedure. On the other hand, the CM method with  $\eta < 1$  requires less stringent initial estimates. (In Table 1,  $\Delta\omega := |\hat{\omega}_n^{(0)} - \omega_0| = 0.18\pi$ .) In fact, the CM estimates obtained with a smaller  $\eta$  can be utilized as initial estimates for a larger  $\eta$ . Eventually, as  $\eta \rightarrow 1$ , one can still achieve the same efficiency of the NLS estimator. In conclusion, the CM method provides a flexible and efficient approach of frequency estimation.

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